ON THE MOTION OF SYSTEMS OF VARYING COMPOSITION IN THE PRESENCE OF VARIATIONAL FORCES

(O DVIZHENII SISTEM PEREMENNOGO SOSTAVA PRI NALICHII VARIATSIONNYKH SIL)

PMM Vol.27, No.3, 1963, pp. 468-473

Iu.E. BOROVSKII and Iu.I. KULAKOV
(Novosibirsk)

(Received October 27, 1962)

In this paper, the equation for the motion of a system of varying composition is obtained, taking into account variational forces arising because of the nonstationary motion of the medium, and connected with the variation of the momentum with respect to a solid shell. The solution of the Okhotsimskii problem is given; and the effect of varying forces on the motion of systems of varying composition with a fluid as an operating substance is studied. The possibility of increasing the final speed of such systems by periodic displacement, by internal forces, of the center of gravity of the system with respect to the solid shell is investigated.

Equations of motion of systems with varying composition are obtained in the most general form in the paper by Gantmakher and Levin [1]. In the paper quoted the general expression for the variational forces is given, arising during nonstationary motion of the medium forming the operating substance, and equal to the variation of the momentum of the system with respect to the solid shell. From this expression it follows that for rockets with liquid or solid fuel the variational forces are negligibly small in comparison to the reactive forces [2].

There exist, however, a number of systems in which these forces play an essential role and taking them into account leads to consequences unusual at first glance.

Let us consider the so-called Okhotsimskii problem.

On a smooth horizontal surface is located a carriage with a tank filled with water. Find the law of motion of the carriage, if the water flows out according to a given law through a vertical tube, situated at a distance l from the center of gravity C_0 of the carriage with the empty vessel.

Let x be the coordinate of the center of gravity C_0 of the carriage with the empty tank, and m_0 its mass. Let, further, the mass of the carriage with the tank containing water be equal to $m(t)$. If, in the process of flowing out, the position of the center of gravity C of the vessel with water is displaced with respect to C_0, then its coordinate x_1 may be written down as

$$x_1 = x + a(t)$$

where $a(t)$ – the coordinate of C with respect to C_0 – is a certain function of time depending on the geometry of the vessel and on the law of efflux. Let us use the law of conservation of the center of gravity of the system consisting of the vessel with the liquid and the drops flowing out of the tube at the moment of time $-\infty < \tau < t$. The static moment of the entire system is

$$(x + a)m + \int_{-\infty}^{t} \xi(t, \tau)\, dm_\tau = MX_c = \text{const} \tag{1}$$

where $\xi(t, \tau)$ is the coordinate at the instant t of the drop of mass dm_τ, discharged from the vessel at the moment τ; M is the mass of the entire system.

If the drop leaves the tube with a relative velocity whose projection onto the x-axis is u, then

$$\xi(t, \tau) = x + l + (x + \dot{l} + u)(t - \tau) \tag{2}$$

It is assumed that, in general, the tube (nozzle) can be displaced with respect to the vessel by means of a certain mechanism, according to a given law $l = l(t)$. Noting that $d\dot{m}_\tau = \dot{m}\,d\tau$, we differentiate (1) twice with respect to t. We obtain

$$m\ddot{x} + m\ddot{a} + \dot{m}(2\dot{a} - 2\dot{l} - u) + \ddot{m}(a - l) = 0 \tag{3}$$

It is evident that equation (3) is applicable not only to the problem of Okhotsimskii but also to reactive systems of the most different design. Let us consider some of them.

1. *Carriage of Okhotsimskii.* For it, l = const, and we shall consider the efflux of the liquid occurring so that the coordinates C and C_0 coincide, i.e. $a = 0$. Since the liquid flows out through the vertical tube, the projection u of the relative velocity onto the x-axis is zero.

Thus equation (3) is reduced to the equation

$$m\ddot{x} = \ddot{m}l \tag{4}$$

from which we easily find the speed of the carriage at any instant t

$$v = l \int_{-\infty}^{t} \frac{\ddot{m}}{m}\, dt \tag{5}$$

Let us consider the discharge regime

$$m = \begin{cases} m_1 & (-\infty < t \leqslant 0) \\ m_1 - \mu t & (0 \leqslant t \leqslant T) \\ m_0 & (T \leqslant t < \infty) \end{cases} \tag{6}$$

We have

$$\ddot{m} = -\mu\delta(t) + \mu\delta(t - T) \tag{7}$$

Substituting (6) and (7) into (5), we obtain*

$$v = \begin{cases} 0 & (-\infty < t \leqslant 0) \\ l\mu / m_1 & (0 < t \leqslant T) \\ lm(1/m_0 - 1/m_1) & (T < t < \infty) \end{cases}$$

Thus, if the tube is situated to the right of the center of gravity C ($l > 0$), then the carriage, beginning at the instant of discharge of the liquid, moves with the constant speed $-l\mu/m_1$ to the left; then, when all the fluid has flowed out of the vessel, the carriage will change the magnitude and direction of its velocity with a jerk, and begin to move to the right with the speed $l\mu(1/m_0 - 1/m_1)$.

2. *Rocket with a fluid as operating substance.* Let the position of the nozzle and the center of gravity C remain invariable with respect to C_0, i.e. l = const and $a = 0$, but the projection onto the x-axis of the relative velocity of the discharged liquid $u \neq 0$. In this case equation (3) will be

$$m\ddot{x} = \dot{m}u + \ddot{m}l \tag{8}$$

Equation (8) is the generalization of the Meshcherskii equation which takes into account the motion of the fluid with respect to the body of the rocket. This equation is easily integrated, so that the speed of the

* It is not difficult to see that v and $\dot{m}$ have the same discontinuities. This is easy to prove by considering the equation obtained from (1) by differentiating once.

rocket at any instant of time turns out to be

$$\dot{x} = u \ln \frac{m}{m_1} + l \frac{\dot{m}}{m} + l \int_{-\infty}^{t} \left(\frac{\dot{m}}{m}\right)^2 dt \tag{9}$$

For the final speed, when $m(\infty) = m_0$, $\dot{m}(\infty) = 0$ we have

$$v = -u \ln \frac{m_1}{m_0} + l \int_{-\infty}^{\infty} \left(\frac{\dot{m}}{m}\right)^2 dt \tag{10}$$

Thus, in contrast to the known result, the taking into account of the relative motion of the fluid leads to the dependence of the final speed of the rocket on the location of the nozzle ($l > 0$ or $l < 0$) and on the regime of discharge $m(t)$.

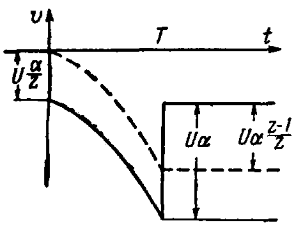

Fig. 1.

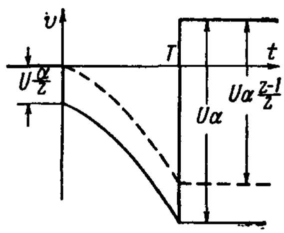

Fig. 2.

Thus, for the regime of discharge (6) the speed of the motion of the rocket turns out to be

$$v = \begin{cases} 0 & (-\infty < t < 0) \\ u \ln [(m_1 - \mu t) / m_1] - l\mu / m_1 & (0 < t < T) \\ u \ln (m_0 / m_1) + l\mu (1 / m_0 - 1 / m_1) & (T < t < \infty) \end{cases}$$

Let us introduce the notations

$$\alpha = \frac{\mu l}{u m_0}, \qquad \zeta = \frac{z \ln z}{z - 1}, \qquad z = \frac{m_1}{m_0} > 1$$

and let us consider the motion of the rocket for various values of α and z.

Let $\alpha > 0$ (nozzle to the right of C):

if $\zeta > \alpha$, then the final velocity of the rocket is directed to the side opposite to the direction of the discharged fluid (Fig. 1); its absolute value turns out to be smaller by $u\alpha(z - 1)/z$ than according to the Tsiolkovskii formula (dashed line in Fig. 1 and 2);

if $\zeta < \alpha$, then the final velocity of the rocket turns out to be directed to the same side to which the fluid is discharged (Fig. 2);

if $\zeta = \alpha$, an unexpected result is obtained: the rocket gathers speed and moves to the side opposite to the direction of the discharged fluid, but at the last moment, when the discharge of the fluid stops, the speed of its motion becomes equal to zero, and the rocket stops at a given space point.

If $\alpha < 0$ (this case is possible when the nozzle is situated to the left of C ($l < 0$)), then for any $z > 1$ the rocket has at the initial instant the speed $u\alpha/z$ directed to the same side to which the fluid is discharged. Then a braking occurs, the velocity changes its sign, and at the last moment the rocket increases its speed with a jump and continues to move in the negative direction with a speed which exceeds that given by the Tsiolkovskii formula by $u\alpha(z - 1)/z$ (Fig. 3).

It can be rigorously shown that for $l < 0$ the discharge regime (6) is the best of all possible regimes $m(t)$ for which $\mu \geqslant -\dot{m}$, in the sense of obtaining the maximal final speed. The proof is obtained from formula (10), if one notes that

$$\left(\frac{\dot{m}}{m}\right)^2 = \left(\frac{1}{m}\right)^{\cdot} (-\dot{m}) \leqslant \left(\frac{1}{m}\right)^{\cdot} \mu$$

From this

$$\int\limits_{\infty}^{\infty} \left(\frac{\dot{m}}{m}\right)^2 dt \leqslant \left(\frac{1}{m_0} - \frac{1}{m_1}\right) \mu$$

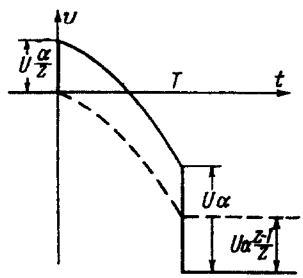

Fig. 3.

Thus the equality sign is obviously valid then, and only then, when $-\dot{m} \equiv \mu$ for $0 < t < T$, i.e. at the discharge regime (6).

Thus, for certain values of the parameters α and z a gain in the final speed of the rocket is possible, as compared with the case $\alpha = 0$, when the Tsiolkovskii formula is valid. It is worth noting that this gain arises because of a more rational redistribution of the energy between the body of the rocket and the ejected fluid, i.e. because of the increase of efficiency

$$\eta = \frac{E^\circ}{E^\circ + E} \quad \left(E = -\frac{1}{2} \int\limits_{-\infty}^{\infty} (u + \dot{x})^2 \dot{m} dt\right)$$

Here E° is the final kinetic energy of the rocket body; E is the kinetic energy of the entire ejected fluid. As simple calculations show, the efficiency turns out to be (for the regime (6))

$$\eta = \left(1 - \frac{1}{z}\right)\left[1 + \frac{z(1-z^{-1})^2 - \ln^2 z}{[\ln z - \alpha(1-z^{-1})]^2}\right]^{-1} \qquad \left(\alpha = \frac{\mu l}{u m_0}, \quad z = \frac{m_1}{m_0}\right) \tag{11}$$

The relation between η and α at constant z is given in Fig. 4.

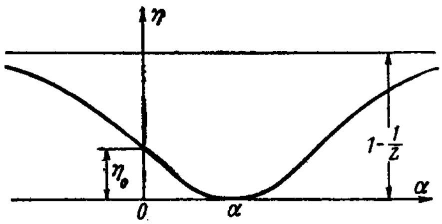

Fig. 4.

The efficiency of the rocket grows with the increase of the nozzle distance from the center of gravity C of the rocket if the nozzle is situated as shown in Fig. 5.

But if the nozzle is located to the right of the center of gravity of the rocket, then for a certain length l, η becomes zero, which corresponds to a stopping of the rocket after ejection of the entire fluid. With further increase of l the efficiency η increases and reaches the limiting value $1 - z^{-1}$ for $\alpha \gg \zeta$. It is noteworthy that the efficiency coefficient according to Tsiolkovskii, η_0, equal to $\eta(0, z)$, has the maximum value $\eta_{max} = 0.65$ for $z \approx 5$, and decreases to zero as $z \to \infty$ (Fig. 6).

3. *Fluid rocket with a motor performing a periodic displacement of the center of gravity of the rocket with respect to its shell.* To represent the essence of the matter more clearly, let us consider an

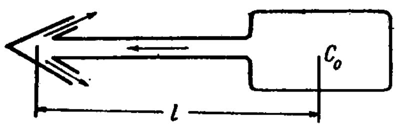

Fig. 5.

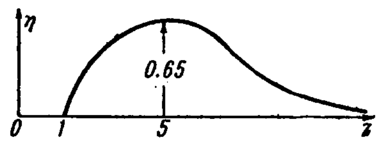

Fig. 6.

ordinary rocket inside which there is a sufficiently large mass which is displaced by internal forces from the front wall to the rear and back.

As a result of such a shifting, the center of gravity C_0 of the body of the rocket will shift with respect to the center of gravity C of the entire rocket, together with the fluid, so that

$$a = a_0 - \delta\varkappa(t)\sin(\omega t + \varphi) \tag{12}$$

where $\kappa(t)$ is a function describing the transition processes at the beginning and the end of discharge; it is definite and twice differentiable in $(-\infty, \infty)$, equal to zero outside the segment $[0, T]$ and equal to unity on $[t_1, t_2]$, where $[t_1, t_2] \subset (0, T)$.

Let $l - a = s$, then for l = const equation (3) becomes

$$\ddot{x} = u\frac{\dot{m}}{m} + \frac{(\ddot{ms})}{m} \tag{13}$$

Thus the speed of the rocket at any instant of time turns out to be

$$v = u\ln\frac{m}{m_1} + \int_{-\infty}^{t} \frac{(\ddot{ms})}{m}\,dt \tag{14}$$

or, integrating by parts

$$v = u\ln\frac{m}{m_1} + \frac{(\dot{ms})}{m} + \int_{-\infty}^{t} \left(\frac{\dot{m}}{m}\right)^2 s\,dt + \int_{-\infty}^{t} \frac{\dot{m}}{m}\,\dot{s}\,dt \tag{15}$$

Let the law of efflux be given in the form

$$m = m_1 \exp\left(-\int_0^t (q + \varepsilon\sin\omega t)\,\varkappa(t)\,dt\right) \qquad (0 < \varepsilon \leqslant q) \tag{16}$$

Noting that $s = s_0 + \delta\kappa(t)\sin(\omega t + \varphi)$, let us make some estimates of the final speed $v(T)$. We have

$$\left(\frac{\dot{m}}{m}\right)^2 s = (q + \varepsilon\sin\omega t)^2 (s_0 + \delta\varkappa(t)\sin(\omega t + \varphi))\,\varkappa^2(t) \tag{17}$$

$$\left(\frac{\dot{m}}{m}\right)\dot{s} = \varkappa(t)(q + \varepsilon\sin\omega t)(\dot{\varkappa}(t)\sin(\omega t + \varphi) + \varkappa(t)\,\omega\cos(\omega t + \varphi))\,\delta \tag{18}$$

In evaluating the integrals of (17) and (18), we use the fact that

$$\int_A^B f(t)\sin(\omega t + \alpha)\,dt \to 0 \qquad \text{as} \quad \omega \to \infty$$

Denoting the sum of such integrals by $\lambda(\omega)$, we obtain for the final speed of the rocket

$$v = u\ln\frac{m_0}{m_1} + \left[\left(q^2 + \frac{\varepsilon^2}{2}\right)s_0 - \frac{1}{2}\delta\omega\varepsilon\sin\varphi\right]A + B\,q\varepsilon\delta\cos\varphi + \\ + C\frac{\delta\varepsilon}{2}\cos\varphi + \lambda(\omega) \tag{19}$$

Here

$$A = \int_0^T \varkappa^2 dt, \qquad B = \int_0^T \varkappa^3 dt, \qquad C = \int_0^T \varkappa\dot{\varkappa} dt = 0$$

Thus, from formula (19) it is seen that, with a sufficiently large frequency ω and $\varphi = \pm 1/2\ \pi$, the rocket's speed may be made very high. By changing the sign of the phase φ one may achieve a change of the direction of the final velocity of the rocket.

Thus, the energy of the internal forces producing a periodic displacement of the center of gravity of the rocket with respect to its shell, may under certain circumstances be used to increase the absolute speed of the rocket. But there is, of course, no contradiction to the basic laws of mechanics: no more takes place than the most rational redistribution of the energy between the rocket body and the ejected fluid, for equality of their moments.

In conclusion we consider it our duty to thank F.R. Gantmakher for his remarks and interest in the paper.

BIBLIOGRAPHY

1. Gantmakher, F.R. and Levin, L.M., Ob uravneniiakh dvizheniia (On equations of motion). *PMM* Vol. 11, No. 3, 1947.

2. Gantmakher, F.R. and Levin, L.M., *Teoriia poleta neupravliaemykh raket (Theory of flight of uncontrollable rockets)*. Fizmatgiz, 1959.

Translated by D.R.M.